

# Strings on Calabi–Yau spaces and Toric Geometry

Maximilian Kreuzer<sup>a\*</sup>

<sup>a</sup>Institut für Theoretische Physik, Technische Universität Wien  
Wiedner Hauptstraße 8–10, A-1040 Wien, AUSTRIA

After a brief introduction into the use of Calabi–Yau varieties in string dualities, and the role of toric geometry in that context, we review the classification of toric Calabi–Yau hypersurfaces and present some results on complete intersections. While no proof of the existence of a finite bound on the Hodge numbers is known, all new data stay inside the familiar range  $h_{11} + h_{12} \leq 502$ .

## 1. INTRODUCTION

Calabi–Yau manifolds are an important ingredient for constructing (quasi-)realistic string models, because  $N = 1$  supersymmetry below the string scale essentially implies a complex structure and the existence of a Ricci-flat Kähler metric on the effective space-time manifold [1,2]. While exact conformal techniques provide an important complementary tool [3,4], geometric compactification has the advantage that we can vary the moduli of a string model and are not stuck on isolated “Gepner points”, where the conformal field theory is rational.

The most important problem of string theory, its lack of a non-perturbative definition, is still unsolved. Nevertheless, some aspects of non-perturbative string physics became accessible in 1995 through the discovery of nonperturbative dualities [5–7]. These allow us to compute certain quantities at strong coupling by relating them to a weak coupling situation in a dual model. These dualities have been tested, for example, by comparing quantities that are protected against non-perturbative corrections by a sufficient amount of supersymmetry.

Dualities exchange elementary degrees of freedom with composite (solitonic) states that become light at strong coupling. In perturbative string theory the light particle states come from string oscillations, Kaluza–Klein excitations, and

winding modes. The latter can be regarded as  $\sigma$  model solitons and lead, for example, to T-duality of torus compactifications [8]. A generalization of this duality to curved space, called mirror symmetry [9,10], operates between topologically distinct Calabi–Yau manifolds by exchanging complex structure and Kähler moduli and allows us to sum up sigma model corrections to certain quantities [11–13].

Contributions to the light particle spectrum that are non-perturbative in the string coupling essentially come from two different sources: Solitonic branes that wrap small cycles of a compact internal manifold become massless in the limit where the volume of the cycle vanishes. The additional massless particles that arise in this way turned out to match the change in the Hodge data that occurs in conifold transitions [14] between Calabi–Yau manifolds of different topologies via singular limits. In type II compactifications the physics of such a transition is smooth, with D-branes wrapped on one side of the singularity turning into elementary particles [15,16]. Since the D-branes can be identified with black supergravity branes [7], this effect was called black hole condensation.

The second type of contribution arises when the background itself is non-perturbative. Then, for example, open strings stretching between nearby D-branes lead to gauge and matter fields that contribute to the low energy effective theory [17–19], but are localized in the internal dimensions (which could become quite large as compared to

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the Planck scale). Witten observed that strongly coupled IIA strings grow an additional effective dimension. The resulting limit of string theory, whose effective low energy theory is eleven-dimensional supergravity, is called M-theory [5]. IIB strings, on the other hand, are selfdual with duality group  $SL(2, \mathbb{Z})$ , whose geometrization led Vafa to propose a twelve-dimensional origin of this duality, called F-theory [20]. In many of the resulting dualities fibration structures play an important role [21,22] and toric geometry provides us with a very general and powerful framework for analyzing the relevant mathematical structures.

We begin with some general comments on constructing Calabi–Yau manifolds. In section 3 we introduce toric geometry and comment on topology change. Then we come to the classification program of reflexive polyhedra, which naturally encode the combinatorial data that define toric Calabi–Yau hypersurfaces. We close with our results and discuss their use for analyzing string dualities.

## 2. HOW TO MAKE A CALABI–YAU

Compact Kähler spaces with  $c_1 = 0$  are usually constructed as (intersections of) hypersurfaces of some simpler compact spaces, like projective space  $\mathbb{P}^n$ , Grassmannians, or some more general coset spaces. These symmetric ambient spaces all have positive Ricci curvature, whose contribution to  $c_1$  must be compensated by the hypersurface equations. The prototype of this construction is the quintic in  $\mathbb{P}^4$ , which is defined as the vanishing locus of a homogeneous polynomial of degree 5. A generalization to complete intersections in products of projective spaces produced 265 different pairs of Betti numbers  $b_2 = h_{11}$  and  $b_3 = 2h_{12} + 2$  (cf. *Fig. 1*) [23,24]. Here  $h_{11}$  counts the Kähler moduli, which mostly correspond to the volumes of the ambient space factors. The complex structure deformations  $h_{12}$ , on the other hand, get a much larger contribution from the parameters in the defining equations. Thus all Euler numbers  $\chi = 2(h_{11} - h_{12})$  turned out negative.

This seemed to be bad news for the idea of mirror symmetry, which exchanges  $h_{11}$  and  $h_{12}$ , but from the conformal field theory point of view

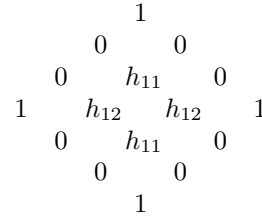


Figure 1. The Hodge diamond of a Calabi–Yau manifold. The Betti numbers are  $b_n = \sum_{p+q=n} h_{pq}$ .

simply corresponds to charge conjugation [9]. Using, however, weighted projective spaces as ambient spaces, the resolution of the singularities that come from the weighted identification

$$(z_1, \dots, z_N) \sim (\lambda^{q_1} z_1, \dots, \lambda^{q_N} z_N) \quad (1)$$

introduces additional Kähler moduli, and indeed, a construction of some 6000 weight systems  $\vec{q} = (q_1, \dots, q_N)$  that admit Calabi–Yau hypersurfaces produced a set of Hodge data that was 90% mirror symmetric [25]. Surprisingly, a complete classification of all 7555 transversal weights made the result less symmetric, with only 83% of the data coming in mirror pairs [26,27], and generalizations like orbifolding [28] and discrete torsion [29] did not improve the situation. In retrospective, this is a consequence of the fact that the orbifolding construction of mirror manifolds discovered by Berglund and Hübsch [30–33] only works for certain subclasses of transversal weights, and many more complicated weights only showed up in the complete lists.

Progress in the understanding of mirror symmetry came via a further generalization of the ambient spaces under consideration to toric varieties. This development is due to Batyrev, who found a manifestly mirror symmetric construction of toric Calabi–Yau spaces for which the mirror map is implemented as a simple combinatorial duality of lattice polyhedra [34].

## 3. TORIC GEOMETRY

The simplest approach to toric varieties [35,36] uses the homogeneous coordinate ring that has

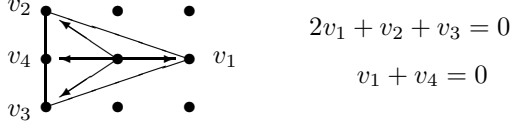


Figure 2. The scaling weights for the coordinates  $z_i$  are  $\vec{q}^{(1)} = (2, 1, 1, 0)$  and  $\vec{q}^{(2)} = (1, 0, 0, 1)$ .  $z_4$  implements the blowup of the singular point  $(1:0:0)$ , which gets excluded from  $W\mathbb{P}_{2,1,1}^2$  by adding  $v_4$ .

been introduced by Cox [37]. This construction is similar to weighted projective space, but with an arbitrary number  $n$  of scaling identifications

$$(z_1, \dots, z_N) \sim (\lambda^{q_1^{(i)}} z_1, \dots, \lambda^{q_N^{(i)}} z_N), \quad i \leq n. \quad (2)$$

The rational scaling weights  $\vec{q}^{(i)}$  can be encoded by linear relations of lattice vectors as shown in *Fig. 2*. In order for the resulting quotient  $(\mathbb{C}^N - Z)/(\mathbb{C}^*)^n$  to be well behaved, it is necessary to subtract a more general exceptional set  $Z$ , which is defined in terms of a collection  $\Sigma$  of cones  $\sigma$  that is closed under intersections (the “algebraic torus”  $\mathbb{C}^*$  is the multiplicative group of non-zero complex numbers).  $\Sigma$  is called the fan of the toric variety and the exceptional set  $Z$  ensures that only coordinates whose respective vectors belong to the same cone can vanish simultaneously.

In the example of *Fig. 2* the vectors  $\{v_1, v_2, v_3\}$  define the weighted projective space  $W\mathbb{P}_{2,1,1}^2$  with scaling weights  $\vec{q} = (2, 1, 1)$ . The exceptional set would only consist of the origin, as any subset of two coordinates belongs to one of the three cones spanned by  $(v_i, v_j)$ . It is easy to see that  $W\mathbb{P}_{2,1,1}^2$  has a  $\mathbb{Z}_2$  quotient singularity at the point with homogeneous coordinates  $(1:0:0)$ . This singularity can be resolved by introducing the additional vector  $v_4$  and the corresponding additional scaling relation.  $v_4$  subdivides the cone  $(v_2, v_3)$  and thereby moves the singular point to the new exceptional set  $Z = \{z_2 = z_3 = 0\} \cup \{z_1 = z_4 = 0\}$ . The singularity gets blown up to a complete  $\mathbb{P}^1$ : Whenever  $z_4 \neq 0$  we can scale it to 1 and recover all of  $W\mathbb{P}_{2,1,1}^2$  except for the point  $z_2 = z_3 = 0$ . For  $z_4 = 0$  we can scale away  $z_1$ , but are left with

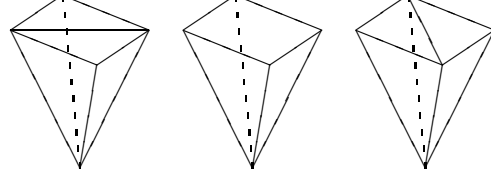


Figure 3. Flop transition: Different triangulations of a cone in the fan lead to topologically distinct desingularizations.

a complete  $\mathbb{P}^1$  that is parametrized by the homogeneous coordinates  $(z_2 : z_3)$ . It can be shown that a toric variety is non-singular if all cones are simplicial and regular (i.e. they are generated by vectors that span a simplex of volume 1 in lattice units) [35]. This is the case for the fan shown in *Fig. 2*, so that the blowup of the singular point desingularizes  $W\mathbb{P}_{2,1,1}^2$ .

The dependency of the exceptional set  $Z$  on the complete fan (and not just on the generators of the 1-dimensional cones, which define the homogeneous coordinates) commences only in 3 dimensions: In *Fig. 3* we observe that a quadratic cone can be triangulated in two different ways, thereby increasing the exceptional set in two different ways and yielding two different blowups of the singularity of the non-simplicial situation. The respective toric varieties are topologically distinct, as they have different intersection numbers, but they turn out to have identical Hodge numbers. Connecting topologically distinct spaces through this relatively mild kind of singularity is called a flop transition, and this was the first setting where it could be shown that the corresponding physics varies smoothly despite of the topology change [12].

#### 4. REFLEXIVE POLYHEDRA

The lattice supporting the cone  $\Sigma$  of a toric variety  $\mathbb{P}_\Sigma$  is usually called the  $N$  lattice. The points  $m$  of the dual lattice  $M = \text{Hom}(N, \mathbb{Z})$  also play an important role, because they correspond to monomials  $\prod z_i^{\langle m, v_i \rangle}$  that provide sections of

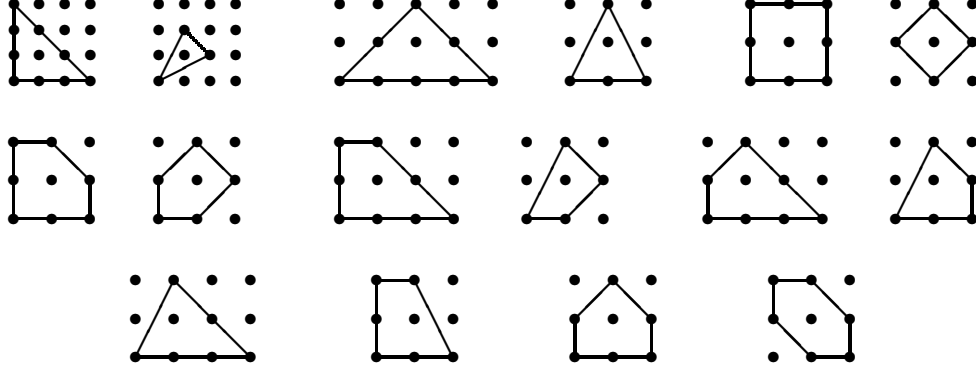


Figure 4. All 16 reflexive polygons in 2D: The first 3 dual pairs are maximal/minimal and contain all others as subpolygons, while the last 4 polygons are selfdual.

certain line bundles over  $\mathbb{P}_\Sigma$ . These line bundles can be constructed in terms of lattice polyhedra  $\Delta \subset M$ . We define the dual of a polyhedron with  $0 \in \Delta$  as

$$\Delta^* = \{y \in N_{\mathbb{R}} \mid \langle y, x \rangle \geq -1 \forall x \in \Delta \subset M_{\mathbb{R}}\}, \quad (3)$$

where  $M_{\mathbb{R}}$  denotes the real extension  $M \otimes_{\mathbb{Z}} \mathbb{R}$  of the lattice  $M$ . The important result of Batyrev was that the generic section of a line bundle that corresponds to  $\Delta$  defines a Calabi–Yau hypersurface in  $\mathbb{P}_\Sigma$  if  $\Delta^*$  is a lattice polytope and if the fan  $\Sigma$  is the fan of cones over the faces of  $\Delta^*$  [34] (to get a smooth Calabi–Yau manifold, this fan still has to be triangulated). Moreover, the Hodge numbers can be computed by a simple combinatorial formula in terms of the numbers of interior points of dual faces of the reflexive pair of polyhedra, and mirror symmetry is simply implemented by exchanging  $\Delta$  and  $\Delta^*$ .

A lattice polyhedron whose dual vertices belong to the dual lattice is called reflexive. This condition is equivalent to  $\Delta$  having exactly one interior point with all facets at distance 1 (i.e. there are no parallel lattice hyperplanes between a facet and the interior point). The classification of toric Calabi–Yau hypersurfaces therefore reduces to the classification of reflexive polyhedra.

The 16 reflexive polygons that exist in two dimensions were first constructed by Batyrev and

are shown in *Fig. 4* in order to illustrate our approach to the classification, which has now been completed in up to 4 dimensions [38–43]. The first step in this program is the identification of a set of objects that contain all others as subpolytopes. In the two-dimensional case these are the three polygons with 10, 8 and 9 points in the first line of *Fig. 4*. Their duals are minimal in the sense that they lose the interior lattice point if one of their vertices is dropped. The linear relations among the vertices of (a simplex decomposition of) these minimal objects are used as our starting point [38,42] and we define the corresponding weight systems via the barycentric coordinates of the origin, i.e.  $\sum q_i v_i = 0$  ( $q_i$  can be chosen to be integer without common divisors).

With  $d = \sum q_i$  this gives an efficient (coordinate independent) description of the dual maximal object by the equation  $\sum q_i n_i = d$  for non-negative integers  $n_i \geq 0$ . If the unique candidate  $n_i = 1$  for an interior point is indeed in the interior of the convex hull we call  $\vec{q}$  an “IP weight system”. The maximal simplexes in *Fig. 4*, for example, correspond to  $n_1 + n_2 + n_3 = 3$  and  $n_1 + n_2 + 2n_3 = 4$ , respectively (the points  $\vec{n} \in \Delta$  correspond to the Newton polyhedron of a quasi-homogeneous polynomial, providing a link to Calabi–Yau hypersurfaces in  $W\mathbb{P}^4$ ). The square, on the other hand, is the convex hull of two one-dimensional simplexes and requires a

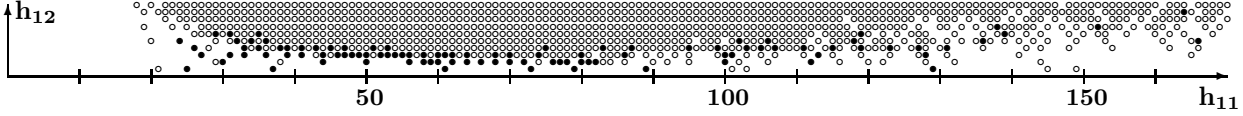


Figure 5. 102 new CICY spectra in the background of hypersurfaces with  $h_{11} \leq 170$  and  $h_{12} \leq 10$ .

combination of the two weight systems  $(1,1,0,0)$  and  $(0,0,1,1)$ . This results in the description of the dual maximal square by  $n_1 + n_2 = n_3 + n_4 = 2$ .

The constructive proof that the number of rational IP weight systems is finite in principle provides an algorithm for the construction of all reflexive polytopes [38,39,42]. To make it work in finite time it still requires a number of refinements that were developed during our implementations for the classification in 3 and 4 dimensions [41,43]. An important subtle point in this context is the fact that the IP weight system fixes a unique coarsest lattice  $N_{\text{coarse}}$ , which is generated by the vertices of the minimal polytope, but leaves open the possibility that the dual maximal object lives on a sublattice of the dual finest relevant lattice  $M_{\text{fine}} = \text{Hom}(N_{\text{coarse}}, \mathbb{Z})$ .

## 5. RESULTS

In three dimensions the 95 IP weight systems have been known for some time. Adding the relevant combined IP weights we found 14 maximal polytopes on  $M_{\text{fine}}$  and one additional maximal polytope on a  $\mathbb{Z}_2$  quotient of the lattice for the quartic in  $\mathbb{P}^3$ . The total number of reflexive polytopes turned out to be 4319 [41].

In four dimensions there are 184026 IP weight systems and we had to begin with 308 maximal reflexive objects on  $M_{\text{fine}}$ . We found 25 additional maximal reflexive polyhedra on sublattices and a total of 473800776 reflexive subpolytopes, which completes the classification in 4 dimensions [43]. The corresponding toric Calabi–Yau hypersurfaces have 30108 different pairs of Hodge numbers. Storing the results requires about 5 GB of disk space. While these cannot be downloaded from the internet in a reasonable time it is possible to search our data base for polytopes with certain properties at our web page [44].

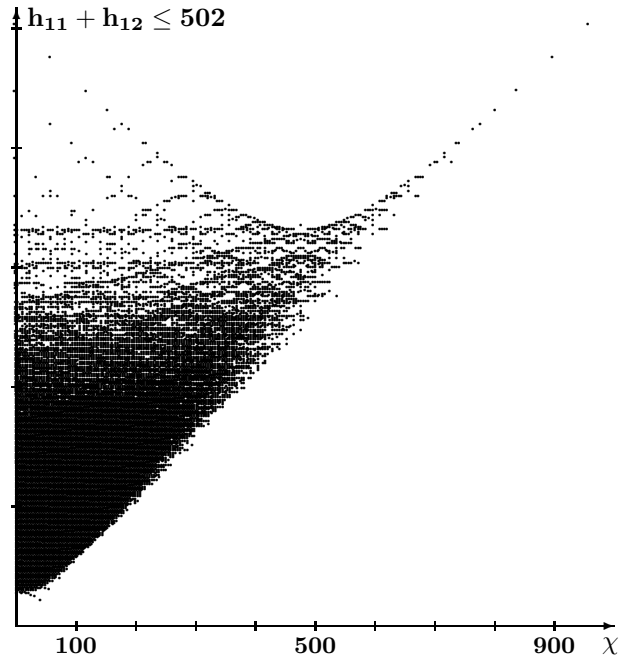


Figure 6. The 15122 (of 30108) hypersurface spectra with  $h_{11} \leq h_{12}$ . The maximal value of  $h_{11} + h_{12}$  comes from  $(251,251)$  and  $(491,11)$ .

A by product of our classification is the proof that all reflexive polytopes in 3 and 4 dimensions are connected by a chain of singular transitions. These correspond to blowing up (or down) a number of divisors that correspond to points of  $\Delta^*$ , which are added (or dropped) to connect the polyhedra. The intermediate singularities are more severe than in the case of flop transitions because also the Euler numbers change. This has been discussed in detail by Strominger et al. in the simplest case of a conifold transition [15,16].

Another important aspect of Calabi–Yau manifolds that is relevant for string dualities are fi-

bration structures. These are encoded in toric geometry in a very transparent way because, for example, elliptic and K3 fibrations show up as reflexive section of  $\Delta^* \subset N$  with dimensions two and three, respectively. A large number of K3 fibrations has been found for the manifolds defined by the 184026 IP weight systems in 4 dimensions [40]. Elliptic fibrations, which are important in F-theory, can be constructed easily with certain types of combined weight systems, as has been discussed in [42,45].

While we have focused on toric hypersurfaces, the mirror involution was extended to complete intersection Calabi–Yau manifolds (CICY) by Batyrev and Borisov [46–49], who again found a very beautiful combinatorial duality in terms of nef partitions, which are certain decompositions of reflexive polyhedra into Minkowski sums. Using these results we analyzed a sizeable number of nef partitions of 5-dimensional polyhedra and computed the resulting Hodge numbers of varieties of codimension 2 [50]. The new spectra that we found are shown in *Fig. 5*. In particular, we doubled the number of known spectra with  $h_{11} = 1$ .

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